# Lecture 1: Kolmogorov's construction of diffusion processes

# Brownian motion

**Definition 1**

$(W_t)_{t \geqslant 0}$ is a stochastic process satisfying:

1. **Stationary independent increments and Gaussian property**: For $t > s$, the increment $W_t - W_s$ follows a normal distribution with mean 0 and variance $(t-s)I_{d\times d}$, and the increment $W_t - W_s$ is independent of the process $(W_u)_{0 \leqslant u \leqslant s}$ before time $s$;
2. **Path continuity**: $(W_t)_{t \geqslant 0}$ is almost surely continuous;

Usually, we assume $W_0 = 0$, in which case, $W$ is called standard Brownian motion.

## Theorem 2 (Kolmogorov's consistency theorem)

*Let $E$ be a standard measure space. Assume that we are given for every $t_1, ..., t_n \in \mathbf{T}$ a probability measure $\mu_{t_1 \cdots t_n}$ on $E^n$, and that these probability measures satisfy:*

(i) *for each $\tau \in S_n$ and $A_i \in \mathcal{E}$,*

$$\mu_{t_1 \cdots t_n}(A_1 \times ... \times A_n) = \mu_{t_{\tau(1)} \cdots t_{\tau(n)}}(A_{\tau(1)} \times ... \times A_{\tau(n)});$$

(ii) *for each $A_i \in \mathcal{E}$,*

$$\mu_{t_1 \cdots t_n}(A_1 \times ... \times A_{n-1} \times E) = \mu_{t_1 \cdots t_{n-1}}(A_1 \times ... \times A_{n-1}).$$

*Then, there is a unique probability measure $\mathbf{P}$ on $(E^{\mathbf{T}}, \mathcal{E}^{\mathbf{T}})$ such that for $t_1, ..., t_n \in \mathbf{T}, A_1, ..., A_n \in \mathcal{E}$:*

$$\mathbf{P}(f(t_1) \in A_1, ..., f(t_n) \in A_n) = \mu_{t_1, ..., t_n}(A_1 \times ... \times A_n).$$

**Theorem 3 (Kolmogorov's continuity theorem)**

*Let $I = [0, T]$, and let $p > 1$ and $\beta \in (1/p, 1)$. Assume $(Y_t \in \mathbb{R}^d)_{t \in I}$ satisfies*

$$\mathbf{E}|Y_s - Y_t|^p \leqslant C|t - s|^{1+\beta p}, \qquad \forall t, s \in I. \tag{1}$$

*Then there exists a version of $Y$, say $X$ (for each $t \in I$, $\mathbf{P}(X_t = Y_t) = 1$), such that*

$$\mathbf{P}\left(\sup_{t \in I} \frac{|X_t - X_s|}{|t - s|^\alpha} \leqslant K\right) = 1,$$

*where $\alpha \in (0, \beta - 1/p)$, $K = K(\alpha, \beta, p, C, I, \omega)$ and $\mathbf{E}K^p < \infty$.*

# Markov processes

## Definition 4

A homogeneous transition function on $(E, \mathcal{E})$ is a collection $P_t,\ t \geqslant 0$ of transition probabilities on $(E, \mathcal{E})$ such that

$$P_{s+t} = P_s P_t, \quad s, t \geqslant 0$$

A process X is Markov with transition function $P = (P_t)_{t \geqslant 0}$, and with respect to a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbf{P})$ if it is adapted and

$$\mathbf{E}\left(f(X_t) \mid \mathcal{F}_s\right) = P_{t-s} f(X_s), \quad t > s.$$

The identity $P_{s+t} = P_s P_t$ is known as the Chapman-Kolmogorov equation.

## Proposition 1

*Let* $(E, \mathcal{E})$ *be a measurable space, and* $\Omega = E^{\mathbb{R}_+}$. *Denote its coordinate process by* $X$,

$$X_t \colon \Omega \to E, \quad \omega \mapsto X_t(\omega) = \omega(t).$$

*Also, let* $\mathcal{F}^0$ *be the* $\sigma$*-algebra generated by* $\{X_t \colon t \in \mathbb{R}_+\}$ *and, for each* $t \geqslant 0$, *let* $\mathcal{F}^0_t$ *be the* $\sigma$*-algebra generated by* $\{X_s \colon s \leqslant t\}$. *So,* $(\mathcal{F}^0_t)_{t \geqslant 0}$ *is a filtration on the measurable space* $(\Omega, \mathcal{F}^0)$ *with respect to which* $X$ *is adapted.*
*Then, for every transition function* $(P_t)_{t \geqslant 0}$ *and probability distribution* $\mu$ *on* $E$, *there is a unique probability measure* $\mathbb{P}$ *on* $(\Omega, \mathcal{F}^0)$ *under which* $X$ *is a Markov process with transition function* $(P_t)_{t \geqslant 0}$ *and initial distribution* $\mu$.

For any times $0 = t_0 < t_1 < \cdots < t_n$ and bounded measurable function $f\colon E^{n+1} \to \mathbb{R}$,

$$\begin{aligned}
&\mathbf{E}[f(X_{t_0}, \dots, X_{t_n})] \\
&= \int\int \cdots \int f(x_0, \dots, x_n)\, P_{t_n - t_{n-1}}(x_{n-1}, \mathrm{d}x_n) \\
&\qquad \cdots P_{t_1 - t_0}(x_0, \mathrm{d}x_1) \mu(\mathrm{d}x_0).
\end{aligned}$$

# Diffusion processes

## Assumption 1

*There exists* $\alpha \in (0,1)$ *and* $\Lambda > 1$ *such that*

$$\Lambda^{-1}|\xi|^2 \leqslant a_{ij}\xi_i\xi_j \leqslant \Lambda|\xi|^2$$

*and*

$$\|a\|_{C^\alpha} = N_1 < \infty, \quad \|b\|_{L^\infty} = N_2 < \infty.$$

Put

$$\mathbb{D} = \{(t,x,y) : 0 \leqslant t \leqslant 1, x, y \in \mathbb{R}^d, x \neq y\}.$$

## Theorem 5 (Heat Kernel estimate I)

*There is a unique continuous function* $p(t, x, y) \in \mathbb{D}$ *such that*

$$\partial_t p(\cdot, y) = Lp(\cdot, y), \quad y \in \mathbb{R}^d.$$

*Moreover,*

(i) *for any* $f \in C_0(\mathbb{R}^d)$, $P_t f \to f$ *uniformly.*

(ii)

$$p \geqslant 0 \quad \textit{and} \quad \int_{\mathbb{R}^d} p(t, x, y) \mathrm{d}y = 1,$$

(iii)

$$p(t + s, x, y) = \int_{\mathbb{R}^d} p(t, x, z) p(s, z, y) \mathrm{d}z$$

(iv)

$$t^{-\frac{d}{2}} \exp(-C|x|^2/t)) \lesssim p(t, x, y) \lesssim t^{-\frac{d}{2}} \exp(-|x|^2/(Ct)))$$